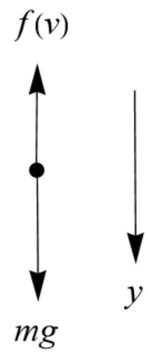
Problem 2.32

Consider the following statement: If at all times during a projectile's flight its speed is much less than the terminal speed, the effects of air resistance are usually very small.
(a) Without reference to the explicit equations for the magnitude of v_{ter}, explain clearly why this is so. **(b)** By examining the explicit formulas (2.26) and (2.53) explain why the statement above is even more useful for the case of quadratic drag than for the linear case. [*Hint:* Express the ratio f/mg of the drag to the weight in terms of the ratio v/v_{ter}.]

Solution

Part (a)

Draw a free-body diagram for a projectile falling down in a medium with air resistance. Let the positive y-direction point downward.



Apply Newton's second law in the y-direction.

$$\sum F_y = ma_y$$

Let $v_y = v$ to simplify the notation.

$$mg - f(v) = m\frac{dv}{dt}$$

The terminal velocity occurs when the velocity reaches equilibrium.

$$mg - f(v_{\text{ter}}) = m(0)$$

As a result,

$$f(v_{\text{ter}}) = mg.$$

Suppose that the projectile's velocity is much less than the terminal velocity.

$$v \ll v_{\text{ter}}$$

Divide both sides by v_{ter}.

$$\frac{v}{v_{\text{ter}}} \ll 1$$

Multiply both sides by mg.

$$\left(\frac{v}{v_{\text{ter}}}\right) mg \ll mg$$

Provided that the projectile's velocity is nowhere near the speed of sound, the drag force function can be expanded in a power series about $v = 0$. This means

$$f(v_{\text{ter}}) = bv_{\text{ter}} + cv_{\text{ter}}^2 + dv_{\text{ter}}^3 + \cdots = mg$$

and

$$\begin{aligned}
f(v) &= bv + cv^2 + dv^3 + \cdots \\
&= bv_{\text{ter}}\left(\frac{v}{v_{\text{ter}}}\right) + cv_{\text{ter}}^2\left(\frac{v}{v_{\text{ter}}}\right)^2 + dv_{\text{ter}}^3\left(\frac{v}{v_{\text{ter}}}\right)^3 + \cdots \\
&< bv_{\text{ter}}\left(\frac{v}{v_{\text{ter}}}\right) + cv_{\text{ter}}^2\left(\frac{v}{v_{\text{ter}}}\right) + dv_{\text{ter}}^3\left(\frac{v}{v_{\text{ter}}}\right) + \cdots \\
&= \left(\frac{v}{v_{\text{ter}}}\right)(bv_{\text{ter}} + cv_{\text{ter}}^2 + dv_{\text{ter}}^3 + \cdots) \\
&= \left(\frac{v}{v_{\text{ter}}}\right) mg \\
&\ll mg.
\end{aligned}$$

Therefore, the drag force is negligible compared to the gravitational force if the projectile's velocity is much less than its terminal velocity.

Part (b)

Equation (2.26) is on page 50 and gives the terminal velocity in a medium with linear air resistance:

$$v_{\text{ter}} = \frac{mg}{b}. \tag{2.26}$$

Equation (2.53) is on page 60 and gives the terminal velocity in a medium with quadratic air resistance:

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}. \tag{2.53}$$

Follow the hint and write out the ratio of drag to weight in each case.

$$\begin{cases} \text{Linear:} & \dfrac{f_{\text{lin}}}{mg} = \dfrac{bv}{mg} = \dfrac{v}{\frac{mg}{b}} = \dfrac{v}{v_{\text{ter}}} \\ \text{Quadratic:} & \dfrac{f_{\text{quad}}}{mg} = \dfrac{cv^2}{mg} = \dfrac{v^2}{\frac{mg}{c}} = \dfrac{v^2}{v_{\text{ter}}^2} = \left(\dfrac{v}{v_{\text{ter}}}\right)^2 \end{cases}$$

If the projectile's velocity is much less than its terminal velocity, then

$$v \ll v_{\text{ter}}$$

$$\frac{v}{v_{\text{ter}}} \ll 1,$$

which means $(v/v_{\text{ter}})^2$ is even smaller to the point of being negligible.

$$\begin{cases} \text{Linear:} & \dfrac{f_{\text{lin}}}{mg} = \dfrac{v}{v_{\text{ter}}} \ll 1 & \rightarrow \quad f_{\text{lin}} \ll mg \\ \text{Quadratic:} & \dfrac{f_{\text{quad}}}{mg} = \left(\dfrac{v}{v_{\text{ter}}}\right)^2 \lll 1 & \rightarrow \quad f_{\text{quad}} \lll mg \end{cases}$$

Consequently, the conclusion of part (a) is more applicable to the quadratic case than it is to the linear case.